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RECENT ADVANCES IN STATISTICAL METHODS
FOR SYSTEM RELIABILITY USING
BERNOULLI SAMPLING OF COMPONENTS

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ABSTRACT

The paper provides a survey of results in statistical inference in systems reliability using Bernoulli sampling of individual components. Particular attention is given to the notion of Buehler optimality and its implementation in such problems. Recent results of the authors on Buehler optimal confidence bounds on the reliability of series and parallel systems are discussed. For series systems, these results employ a generalization of an inequality of Sudakov. For parallel systems, Buehler optimal bounds are obtained for small numbers of failures using the notion of Schur concavity. Estimates of the optimal bounds are obtained in those cases for which the property of Schur concavity does not hold.

SIGNiFICANCE AND EXPLANATION

Parallel and series systems arise naturally in practice in engineering and physics. Therefore it is of substantial significance to be able to utilize efficiently data obtained on individual components for the purpose of obtaining an overall assessment of the reliability of the system. The methods of this paper may be employed for this purpose.

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RECENT ADVANCES IN STATISTICAL METHODS FOR SYSTEM
RELIABILITY USING BERNOULLI SAMPLING OF COMPONENTS

by

Bernard Harris[*] and Andrew P. Soms[**]

1. INTRODUCTION

In this paper we examine the history of statistical inference in systems reliability using Bernoulli sampling of components with particular emphasis on the notion of Buehler optimality and its role in such problems. In particular, we focus on recent results of the authors which facilitate obtaining Buehler optimal bounds on the reliability of series and parallel systems. For series systems, these results employ a generalization of an inequality of Sudakov and for parallel systems, Buehler optimal bounds are obtained using the notion of Schur concavity when the number of failures is small. Estimates of the optimal bounds for parallel systems are obtained for those cases in which the technique employing Schur concavity fails.

We suppose that the system has k components and let p_i, $i = 1, 2, \ldots, k$ be the probability that the ith component does not fail. We further assume that the components are stochastically independent. Let $h(p_1, p_2, \ldots, p_k)$ be the probability that the system does not fail. Then the problem under discussion may be described as follows. The experimenter takes N_i observations on each of the k components and records y_i, the number of failures observed on the ith component. Then given this data, we wish to obtain a $1 - \alpha$ lower confidence limit on $h(p_1, p_2, \ldots, p_k)$.

Throughout the discussion, we restrict our attention to coherent systems. These may be described as systems in which the system fails when all components fail and the system functions when all components function. In addition, replacing a defective component by a good component will not cause a functioning system to fail. The reader is referred to R. E. Barlow and F. Proschan [1] for relevant details concerning the reliability function $h(p_1,\ldots,p_k)$ and properties of coherent systems.

The problem discussed herein arises naturally in many situations relevant to acquisition decisions. For such an illustration consider a "one-shot device", namely, one which is to perform a specific function at a specific time, so that the lifetime of the device is not a consideration. Then, you are interested in assuring that the probability that the device will function properly when it is to be used is at least a specified number.

For a second application, consider a system which is to function for a specified length of time, known as the mission time. If the only data available is the number of failures of each component before the mission time, then the model of this paper is appropriate.

Throughout the paper, we emphasize the specific cases of series and parallel systems.

2. BUEHLER OPTIMAL CONFIDENCE LIMITS FOR SYSTEM RELIABILITY

We now introduce the notion of Buehler optimality with respect to the ordering function $g(\tilde{x})$, where $\tilde{x} = (x_1, x_2, \ldots, x_k)$, $x_i = N_i - y_i$, $i = 1, 2, \ldots, k$. In Buehler (1957), the following principle was proposed and applied to obtaining confidence limits for the product of two binomial parameters. This corresponds to proposing an optimal solution to the problem of determining lower confidence limits for the reliability of series systems of two components.

$g(\tilde{x})$ is an ordering function if for $\tilde{x}^{(1)} = (x_{11}, x_{12}, \ldots, x_{1k})$ and $\tilde{x}^{(2)} = (x_{21}, x_{22}, \ldots, x_{2k})$ with $x_{1i} \leq x_{2i}$, $i = 1, 2, \ldots, k$, we have $g(\tilde{x}^{(1)}) \leq g(\tilde{x}^{(2)})$. In the original formulation, one orders the $\prod_{i=1}^{k}(N_i+1)$ sample outcomes and lists the corresponding values of $g(\tilde{x})$. Then calculate

$$P_{\tilde{p}}\{g(x) \geq u\} . \qquad (1)$$

This is the probability, for fixed $\tilde{p}$, of getting an outcome as good as or better than u since $g(\tilde{x})$ is ordered consistently with the number of functioning components observed in the sample.

If we let

$$Q_\alpha(u) = \{\tilde{p}: P_{\tilde{p}}(g(\tilde{x}) \geq u)\} \geq \alpha . \qquad (2)$$

Then Q_α is a $1-\alpha$ confidence region for $\tilde{p}$ determined by $g(\tilde{x})$. This can be extended to a $1-\alpha$ lower confidence bound for $h(\tilde{p})$ by letting

$$\underline{h}(\tilde{u}) = \inf \{h(\tilde{p}): P_{\tilde{p}}(g(\tilde{x}) \geq u) \geq \alpha\} , \qquad (3)$$

for $0 < \alpha < 1$. It can be shown that if $\underline{\ell}(u)$ is any other system of lower confidence limits based on the ordering function $g(\tilde{x})$, then $\underline{\ell}(u) \leq \underline{h}(u)$ for all u. Further details on this optimality property and a proof of the above assertion may be found in Harris and Soms [8].

Unfortunately, while this does give a procedure for obtaining best lower confidence bounds once $g(\tilde{x})$ has been specified, this does not say how to choose $g(\tilde{x})$. For various kinds of systems, many researchers have proposed different ordering functions and quite a few statisticians have suggested procedures which have the Buehler optimality property asymptotically but not necessarily for any fixed sample size. In addition the literature contains many suggested procedures which are not optimal but possess some other favorable attribute such as ease of computation, while not deviating too far from optimality. As originally described by Buehler many procedures employing Buehler optimality are virtually incomputable for $k > 2$.

Other properties which some writers have considered desirable are exactness and conservatism. Specifically, a confidence interval procedure is exact if there are parameter points such that for those points, $\geq \alpha$ in (3) can be replaced by equality. In the case of (3), this is true for a large class of ordering functions, since $h(\tilde{p})$ is continuous and the infimum will be attained. A confidence interval procedure is conservative if the true confidence coefficient is at least $1-\alpha$ for all parameter points. Clearly, many asymptotic approximations and other approximate procedures will violate this.

We discuss this by looking at some specific types of systems, beginning with series systems.

3. LOWER CONFIDENCE LIMITS FCR THE RELIABILITY OF SERIES SYSTEMS

A natural first attempt in selecting an ordering function for series systems is to choose a point estimator for $h(\bar{p}) = \prod_{i=1}^{k} p_i$. Such an estimator is the maximum likelihood estimator given by

$$h(\hat{p}) = \prod_{i=1}^{k} \frac{X_i}{N_i} . \tag{4}$$

In the situation where the p_i's are close to unity, the high reliability case, we can write

$$h(\hat{p}) = \prod_{i=1}^{k} \frac{N_i - Y_i}{N_i} \tag{5}$$

and if the N_i's are large, this is commonly approximated by

$$h(\hat{p}) = \prod_{i=1}^{k} (1 - \frac{Y_i}{N_i}) \sim 1 - \sum_{i=1}^{k} \frac{Y_i}{N_i} . \tag{6}$$

Then, under the assumption that the p_i's are close to unity, if the N_i's are all "large", this suggests replacing the original assumption of the binomial distribution by the Poisson distribution and this alternative has been exploited by many practitioners. In particular, $1-h(\hat{p})$ is regarded as having a Poisson distribution with parameter $\Sigma \lambda_i / N_i$.

Some alternative ordering functions that have been considered are

$$\sum a_i Y_i + z_\alpha (a_i^2 Y_i)^{1/2}, \tag{7}$$

where z_α satisfies $1 - \Phi(z_\alpha) = \alpha$, $\Phi(x)$ is the standard normal distribution function, $a_1 \geq \cdots \geq a_n$ and $a_i = (n_i \sum \frac{1}{n_i})^{-1}$. This choice was proposed by M. V. Johns, Jr. [10] and is somewhat suggestive of (6) in that it is a weighted normal deviate suggested by the Poisson approximation noted previously.

Buehler [2] ordered by computing separate $1 - (1-\alpha)^{1/k}$ confidence bounds. The product of the separate confidence limits is the final lower confidence limit.

It should be noted that Buehler directed most of his discussion to parallel systems and all of his approximations and numerical examples are appropriate only for that application. However, using the duality noted later, the general discussion in his paper can be applied equally well to both series and parallel systems.

Closely related to the maximum likelihood estimator and sometimes employed as an ordering function are estimators of the form

$$g(\tilde{x}) = \prod_{i=1}^{k} (X_i - \alpha_i)/(N_i - \beta_i) , \qquad (8)$$

where α_i, β_i may be dependent on the X_i's and N_i's. Naturally, α_i and β_i must satisfy conditions such that $g(\tilde{x})$ will be asymptotically equivalent to the maximum likelihood estimator (5).

An example of this approach is given by A. Madansky [13], who calculated the Wilks' likelihood ratio $L(\tilde{x})$ and used the asymptotic result that $-2 \log L(\tilde{x})$ has asymptotically a chi-square distribution with one degree of freedom to obtain a lower confidence limit to the reliability function $h(\hat{p})$. He compared the results obtained using the likelihood ratio to those that would be obtained employing the maximum likelihood estimator, where the distribution of the maximum likelihood estimator is approximated by employing asymptotic normality and the usual asymptotic variance for the maximum likelihood estimator. Madansky refers to this as the linearization method.

For this case, the approximate lower confidence limit has the form

$$\Pi \frac{X_i}{N_i} - (q(x))^{1/2} z_\alpha , \qquad (9)$$

where

$$q(x) = \sum_i (\frac{\partial h(\hat{p})}{\partial \hat{p}_i})^2 \hat{p}_i (1 - \hat{p}_i) . \qquad (10)$$

For both of these techniques, the convergence to the asymptotic distribution is not uniform and substantially larger sample sizes are required near the boundary of the parameter space in order for the limiting distribution to provide a satisfactory approximation.

Some technical improvements using the likelihood ratio technique are given in C. Mack [12], but their relationship with the likelihood ratio procedure is not identified in that paper.

The Lindstrom-Madden method, as described in Lloyd and Lipow [11] is of fundamental importance to this discussion. Let $h(\hat{p})$ denote the maximum likelihood estimator as in (5) and let $\underline{N} = \min_{1 \le i \le k} N_i$. Then regard $\underline{N}(1-h(\hat{p})) = V$ as the number of failures in $\underline{N}$ Bernoulli trial and use the usual method of obtaining a lower confidence limit for a single binomial proportion. In general, V will not be an integer, so that the interpretation as a binomial random variable is not completely justified. One proceeds by interpolation in either the tables of the binomial distribution or in the tables of the incomplete beta function. Some forms of non-linear interpolation have also been proposed and investigated.

Although this procedure is widely used by engineers, it is our impression that statisticians tended to view it with some skepticism. However, extensive simulations and numerical computations made prior to the issuance of the Handbook for the Calculation of Lower Statistical Confidence Bounds on System Reliability [3] suggested that it performs as well as any of the competing

methods under investigation in the region of high reliability. In addition the simulations strongtly suggested that it met the requirement of being conservative, that is, that the true confidence level is always at least as high as the nominal confidence level. A proof of this fact using as inequality due to Sudakov [21] is given in Harris and Soms [8]. The Lindstrom-Madden technique has been recommended for series systems in the above mentioned handbook.

Closely related to the Lindstrom-Madden method is the method of "key test results" introduced by K. A. Weaver [22] and extended by A. Winterbottom [23]. The maximum likelihood estimate $h(\hat{p})$ is calculated. Let $\underline{N} = \min_i N_i$. Calculate $[\underline{N}h(\hat{p})]$. This of course is always an integer. Then find the usual binomial confidence limit for N and $[Nh(\hat{p})] = x$. If $Nh(\hat{p})$ is an integer, then the Lindstrom-Madden method and the key test method coincide. Since we have shown [8] that the Lindstrom-Madden method is conservative. It follows that the method of key test results is at least as conservative.

Easterling's [4] modified maximum likelihood method also employs the maximum likelihood estimator. From the usual asymptotic theory for maximum likelihood estimation, the estimator $h(\hat{p})$ has an asymptotic variance given by

$$\sigma^2_{h(p)} = \sum_{i=1}^{k} \left(\frac{\partial h(p)}{\partial p_i}\right)^2 \text{Var}(\hat{p}_i) \ . \tag{11}$$

Replace p_i by $\hat{p}_i$ in the above, thereby obtaining (10), and set

$$\hat{\sigma}^2_{h(p)} = \frac{h(\hat{p})(1-h(\hat{p}))}{\hat{n}} \ . \tag{12}$$

Then regard $\hat{n}$ as the binomial sample size with $\hat{n} h(\hat{p})$ successes observed and obtain the lower confidence interval for a single binomial parameter, interpolating as in the Lindstrom-Madden method.

A further modification is suggested in which $\hat{n}$ and $\hat{x}$ are replaced by the next integer. This is designated as the MMLI method. The suggestion that this may be satisfactory is deduced from an examination of the deficiencies noted in the linearization (maximum likelihood) method by Madansky.

In a related investigation, J. L. Epstein [5] considered the problem of confidence sets for the product of two binomial parameters. This can be interpreted as either a parallel or a series system in the reliability context. Epstein was motivated by some biomedical applications in which the assumption of high reliability for each of the components is not as natural as it is for the engineering reliability context.

In studying this question, Epstein considered the two ordering functions

$$g_1(\tilde{x}) = \frac{X_1 X_2}{N_1 N_2} \, , \quad g_2(\tilde{x}) = \frac{(X_1+1)(X_2+1)}{N_1 N_2} \, .$$

He concluded that the second was preferable to the first, since the partition of the sample space induced by the second is finer than that induced by the first. This effect is particularly pronounced when either X_1 or X_2 is zero. This is not,however, a significant factor for series systems with high reliability. It is, nevertheless, quite important for parallel systems with high reliability, as we shall subsequently observe.

Apparently ideas similar to those deduced by Epstein motivated the development of the technique suggested by Sandia Corporation and referred to as CONLIM [6].

Let $h(\tilde{p})$ be the reliability function of any coherent system. Then let

$$g(\tilde{x}) = h(\hat{p}) \, , \tag{13}$$

where $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_k)$ and

$$\hat{p}_i = (X_i+1)/(N_i+2), \quad 1 \le i \le k \; . \tag{14}$$

For the series system, this reduces to

$$g(\tilde{X}) = \prod_{t=1}^{k} (X_i+1)/(N_i+2). \tag{15}$$

The particular choice of $\hat{p}_i$ leads one to suspect that this was motivated by Bayesian considerations employing independent uniform priors on $0 \le p_i \le 1$. The computer program which uses (13) to calculate lower confidence bounds for system reliability is capable of dealing with a large variety of systems, but unfortunately, a substantial amount of computer time is often required in order to calculate the lower confidence limit.

The Poisson approximation methods, alluded to earlier, have principally been exploited by statisticians from the Soviet Union.

In particular, Pa lov [19] used the ordering function (6), by defining the parameter for which the confidence limit is sought as

$$\rho = \frac{\lambda_1}{N_1} + \frac{\lambda_2}{N_2} + \cdots + \frac{\lambda_n}{N_K} \; . \tag{16}$$

Then, choose M so that $MN_1, \cdots, MN_k$ are all approximately integers. This results in

$$M\rho = M_1\lambda_1 + M_2\lambda_2 + \cdots + M_k\lambda_k \tag{17}$$

and $\Sigma \, M_i Y_i$ is a suitable statistic for estimation of the parametric function (17) and therefore can be easily employed to get confidence bounds on ρ .

As a specific illustration, consider $k = 3$, $N_1 = 600$, $N_2 = 300$, $N_3 = 100$. Then, from (6) ,

$$h(\tilde{p}) \sim 1 - \frac{q_1}{N_1} - \frac{q_2}{N_2} - \frac{q_3}{N_3}$$

and

$$600(1 - h(\tilde{p})) \sim 600 \left(\frac{\lambda_1}{600} + \frac{\lambda_2}{300} + \frac{\lambda_3}{100} \right) \quad ,$$

so that $Y_1 + 2Y_2 + 6Y_3$ is an appropriate statistic to use for estimation of $h(\tilde{p})$ and for other inferential purposes.

Another method for employing Poisson approximations has been given by Mirniy and Soloviev [18]. This technique may be described as follows. Let

$$h(\tilde{p}) = \prod_{i=1}^{k} p_i = e^{\sum_{i=1}^{k} \log p_i} \quad . \tag{18}$$

For p_i near one, as is appropriate for high reliability,

$$\log p_i \sim - (1 - p_i) .$$

Let $\lambda_i = N_i(1-p_i)$, $i = 1,2,\cdots,k$. Then construct an upper confidence limit for $\sum_{i=1}^{n} \lambda_i$ and an approximate upper confidence limit for

$$e^{-\sum_{i=1}^{k}(1-p_i)} \sim e^{-\sum_{i=1}^{k} \lambda_i/N_i} \tag{19}$$

is given by $\max(-\Sigma \lambda_i/N_i)$, where the maximum is over the set $\{ \sum_{i=1}^{k} \lambda_i \leq \Delta \}$, Δ the upper confidence limit referred to above. This procedure is completely analogous to the procedure using (2) and (37), except that an upper confidence bound is being constructed for the unreliability.

This brief survey makes no pretense of being complete. In particular, the reader is referred to the paper by Joan Rosenblatt [20], which contains a survey of many of the early techniques employed for this problem. Likewise, the book by N. Mann, R. E. Schafer, and N. D. Singpurwalla [14] should be

noted. In particular Chapter 10 is relevant to this discussion and a commonly employed method, called the AO method is described therein. Also Bayesian methods- such as these proposed by D. Mastran [16] and D. Mastran and N. D. Singpurwalla [17], have been omitted, since these are obtained using a different set of principles than the methods which are compared in this paper.

Given this large number of available techniques, how does one make a selection? To answer this question, in the course of the preparation of the Handbook for the Calculation of Lower Statistical Confidence Bounds on System Reliability [3], an extensive investigation of various procedures was carried out. This investigation employed both simulation methods and numerical calculations. Among the requirements sought for were conservatism of procedures. However, while procedures that were found to be optimistic (non-conservative) were regarded as not satisfactory, attention was paid as well to accuracy, namely, that the true confidence coefficient should be as close to $(1-\alpha)$, the nominal confidence coefficient, as possible. Ease of computation was also considered. Finally, performance of the statistical procedures for high reliability was regarded as being of much more significance than performance at low reliabilities.

This investigation established among other things, that the AO method and Easterling's two proposals are non-conservative. CONLIM satisfied every require-ment with the exception of ease of computation.

A surprising conclusion of this investigation was that the Lindstrom-Madden method was found to be conservative, but not excessively conservative. In short, it was the opinion of the committee entrusted with the preparation of this handbook, that the Lindstrom-Madden method should be adopted for use and be the method recommended in the handbook.

Since the investigation undertaken in preparation of the handbook employed simulation and the computation of a large number of numerical examples, a theoretical explanation for the impressive performance of the Lindstrom-Madden method is needed.

The answer to this is provided in Harris and Soms [8] and depends on the application of an inequality of Sudakov [21]. The description of these results follow.

Let

$$I_p(r,s) = \frac{1}{B(r,s)} \int_0^p t^{r-1}(1-t)^{s-1}dt .$$

Then, it is well known that

$$\sum_{i=1}^y \binom{n}{i} p^{n-i} q^i = I_p(n-y,y+1). \tag{20}$$

Let $u(n,y,\alpha)$ satisfy

$$\alpha = I_{u(n,y,\alpha)}(n-y,y+1) \tag{21}$$

and assume that $N_1 \le N_2 \le \ldots \le N_k$. Let

$$g(\tilde{X}) = \prod_{i=1}^k (\tilde{X}_i/N_i),$$

and let

$$y_1 = N_1 q_0 = N_1(1 - \prod_{i=1}^k \frac{x_{0i}}{N_i}) , \tag{22}$$

where x_{0i} is the observed value of X_i. Then Sudakov showed that

$$u(N_1,y_1,\alpha) \le b \le u(N_1,[y_1],\alpha), \tag{23}$$

where b is the $1-\alpha$ Buehler optimal lower confidence limit for $g(\tilde{X})$. Now $u(N_1, y_1, \alpha)$ is the Lindstrom-Madden solution. Thus, it follows that if y is an integer, the Lindstrom-Madden solution is optimal.

In Harris and Soms, a generalization of the Sudakov inequality was employed to show that (23) holds when y_1 is computed for any of a large class of ordering functions in addition to the maximum likelihood estimator. This class includes

$$g(\tilde{X}) = 1 - \sum_{i=1}^{k} \frac{Y_i}{N_i} ,$$

$$g(\tilde{X}) = \Pi(X_i + \alpha_i), \quad \alpha_i \geq 0,$$

and $n_{i+1}\alpha_i \geq \alpha_{i+1} n_i$, and the choice (7) employed by Johns. In addition, an improvement on the upper bound in (23) was obtained. In short, if

$$y_i^* = \max \{y_i : g(\tilde{y}) \leq g(\tilde{y}_0), \ y_i = N_i - x_i, \ y_{0i} = N_i - x_{0i}\} \tag{24}$$

and $\tilde{\tilde{y}} = (0, \ldots, 0, y_i, \ldots, y_k)$, $y_1^* = y_1$, then

$$u(N_1, y_1, \alpha) \leq b \leq \min_i u(N_i[y_i^*], \alpha), \tag{25}$$

holds for a class of ordering functions, which includes many commonly in use. For the relevant details, see Harris and Soms. The lefthand side of (25) may be regarded as a generalization of the Linstrom-Madden method.

4. LOWER CONFIDENCE UNITS FOR THE RELIABILITY OF PARALLEL SYSTEMS

To some extent, there is a duality between series and parallel systems. For a parallel system,

$$1-h(\bar{p}) = \prod_{i=1}^{k} q_i = \prod_{i=1}^{k} (1-p) . \qquad (26)$$

Thus, interchanging reliability and unreliability, success and failure, p_i and $1-p_i=q_i$ converts any probability statement about a series system into an "equivalent" statement about a parallel system. Unfortunately, this duality will not extend as readily to the statistical inference aspects. The reasons are that the above duality will result in conservative confidence bounds being changed into non-conservative bounds. Also high reliability becomes high unreliability, an area of little interest to practitioners. Fruther, the Poisson approximation techniques employed in the study of one type of system will not carry over to the other, since small failure probabilities become small success probabilities, again a situation of virtually no interest.

Nevertheless, some parts of the previous discussion for series systems does carry over to parallel systems.

For this reason, it is convenient to replace the problem of finding a lower confidence limit for the reliability by the problem of finding an upper confidence level for the unreliability. Then, to make the analogy with series systems clearer, we will denote the unreliability by $\bar{h}(\bar{p})$ and the ordering function based on failures by $\bar{g}(\bar{X})$, where $\bar{X}$ denote failures, that is

$$1-h(\bar{p}) = \bar{h}(\bar{p}), \quad N_i-X_i = \bar{X}_i ,$$

and $\bar{g}(\bar{X})$ will often be an estimator of the unreliability.

Buehler's [2] method of computing separate $1-(1-\alpha)^{1/k}$ confidence bounds carries over, however, the Poisson approximations that he uses to get specific numerical values are valid only for parallel systems.

Madansky's [13] method using the Wilk's likelihood ratio statistic is equally valid for parallel systems, since in employing a continous approximation, namely the chi-square distribution, the transference from conservatism to optimism does not take place. Thus, all comments about the use of the technique for series systems apply essentially unchanged for parallel systems. Similarly, the comments about use of the asymptotic theory for the maximum likelihood estimator remain unchanged.

The observations of Epstein [5] that the ordering functions $(\bar{X}_1+1)(\bar{X}_2+1)/N_1 N_2$ is preferable to $\bar{X}_1 \bar{X}_2/N_1 N_2$ is far more relevant to parallel systems than to series systems. To see this, note that the maximum likelihood estimator of $\prod_{i=1}^{k} q_i$ is $\prod_{i=1}^{k} \bar{X}_i / \prod_{i=1}^{k} N_i$, where X_i is the number of observed failures of the ith component. In the case of high reliability, $\bar{X}_i$ will tend to be small and one would like to believe that it will frequently be zero. The difficulty with an ordering function such as $\prod_{i=1}^{k} (\bar{X}_i/N_i)$ is that for $k = 2$, the sample outcomes $\bar{X}_1 = 0$, $\bar{X}_2 = 0$ is not distinguished from $\bar{X}_1 = 0$, $\bar{X}_2 = 100$, when $N_1 = N_2 = 100$. More precisely, the partition induced by $\bar{g}(\bar{X})$ is then very coarse when at least one $\bar{X}_i$ is zero. Consequently, the use of a finer partition for the high reliability situation may produce less conservatism and greater accuracy. Presumably, similar considerations motivated the choice of $\hat{p}_i$ for the CONLIM method, which in this case gives

$$\bar{g}(\bar{X}) = \prod_{i=1}^{k} (\bar{X}_i+1)/(N_i+2) . \tag{27}$$

A somewhat different technique for obtaining an upper confidence limit on $\bar{h}(\bar{p})$ is given in Harris [7], where the exponential family theory is used to

obtain the uniformly most accurate unbiased upper confidence limit. However, this technique does not have a simple description in terms of an ordering function.

The specific question of Buehler optimality for parallel systems was considered in Harris and Soms [9]. There the following results were obtained.

Let

$$\bar{g}(\bar{X}) = \prod_{i=1}^{k} (\bar{X}_i + d), \quad 1 < d < 1.5. \tag{28}$$

Then replacing the binomial distribution by the Poisson approximation, for $\tilde{z}$ any failure vector with $x_i = 5$ for some $1 \le i \le k$ and $x_i = 0$ otherwise, and letting

$$A_{\tilde{z}} = \{\tilde{x}_0 : g(\tilde{x}_0) \le g(\tilde{z})\} \tag{29}$$

where

$$P_{\tilde{\lambda}}\{g(\bar{X}) \le g(\bar{x}_0)\}$$

is a Schur-concave function of $- \ln \lambda_i$, $i = 1,2,\ldots,k$, where $\tilde{\lambda}$ is the parameter vector for the Poisson approximation obtained by letting $\lambda_i = N_i \bar{p}_i$, $i = 1,2,\ldots,k$). The reader is referred to A. W. Marshall and I. Olkin [15] for details on the properties of Schur-concave functions. This enables one to conclude that for $\tilde{x}_0 \in A_z$ the Buehler optimal upper confidence limit for $\prod_{i=1}^{k} \lambda_i$ is given by the solution of

$$P_{\tilde{\lambda}*}\{g(\tilde{X}) \le g(\tilde{x}_0)\} = \alpha , \tag{30}$$

where $\tilde{\lambda}^* = (a,\ldots,a)$, namely has all components identical. This Schur-concavity argument may be extended to other ordering functions than (28). In

particular, the choice used by Buehler [2] also satisfies the necessary conditions.

Unfortunately, if the observed vector of failures lies outside of $A_{\bar{z}}$, this conclusion does not hold for the ordering function (23). Here, upper and lower bounds for the Buehler optimal upper confidence limit have been constructed. The reader is referred to the original paper for details. [9].

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The paper provides a survey of results in statistical inference in systems reliability using Bernoulli sampling of individual components. Particular attention is given to the notion of Buehler optimality and its implementation in such problems. Recent results of the authors on Buehler optimal confidence bounds on the reliability of series and parallel systems are discussed. For series systems, these results employ a generalization of an inequality of Sudakov. For parallel systems, Buehler optimal bounds are obtained for small numbers of failures using the notion of Schur concavity. Estimates of the optimal bounds are obtained in those cases for which the property of Schur concavity does not hold.

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